Chapter 14 Differentiation 2

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1. Variables x and y are related by the equation $y = \frac{\ln x}{e^x}$.

   a. Show that $\frac{dy}{dx} = \frac{1 - x \ln x}{x e^x}$.

$y = \frac{\ln x}{e^x}$

[4]

$\frac{dy}{dx} = \frac{\frac{1}{x} \times e^x - e^x \ln x}{(e^x)^2}$

$= \frac{\cancel{e^x}\left(\frac{1}{x} - \ln x\right)}{(e^x)^{\cancel{2}\, 1}}$

$= \left(\frac{1}{x} - \ln x\right) \times \frac{1}{e^x}$

$= \frac{1 - x\ln x}{x} \times \frac{1}{e^x}$

$= \frac{1 - x \ln x}{x e^x}$ (shown)

   b. Hence find the approximate change in y as x increases from 2 to $2 + h$, where h is small.

$y' = \frac{1 - x\ln x}{x e^x}$ $x = 2$, $\delta x = h$

[2]

$\frac{dy}{dx} = \frac{1 - 2\ln 2}{2e^2} = -0.0261$

$\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$

$-0.0261 \approx \frac{\delta y}{h}$

$\delta y \approx -0.0261h$

2. The number, *B*, of a certain type of bacteria at time *t* days can be described by $B = 200e^{2t} + 800e^{-2t}$.

   a. Find the value of *B* when $t = 0$. [1]

   $B = 200 + 800$

   $= 1000$

   b. At the instant when $\frac{dB}{dt} = 1200$, show that $e^{4t} - 3e^{2t} - 4 = 0$. [3]

   $\frac{dB}{dt} = 400e^{2t} - 1600e^{-2t}$

   $400e^{2t} - 1600e^{-2t} = 1200$

   $(\div 400)$

   $e^{2t} - 4e^{-2t} = 3$

   $e^{2t} - \frac{4}{e^{2t}} = 3$

   $(\times e^{2t})$

   $e^{4t} - 4 - 3e^{2t} = 0$

   $e^{4t} - 3e^{2t} - 4 = 0$ (shown)

   c. Using the substitution $u = e^{2t}$, or otherwise, solve $e^{4t} - 3e^{2t} - 4 = 0$. [2]

   $u^2 - 3u - 4 = 0$

   $(u-4)(u+1) = 0$

   $u = 4$ or $u = -1$

   $e^{2t} = 4$ $\quad$ $e^{2t} = -1$ (reject)

   $2t = \ln 4$

   $t = \frac{1}{2}\ln 4$

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3. It is given that $y = \frac{ln(2x^3+5)}{x-1}$ for $x > 1$.

a. Find the value of $\frac{dy}{dx}$ when $x = 2$. You must show all your working.

[4]

$$y' = \frac{\frac{1}{2x^3+5} \times 6x^2 (x-1) - \ln(2x^3+5)}{(x-1)^2}$$

$$= \frac{\frac{6x^3-6x^2}{2x^3+5} - \ln(2x^3+5)}{(x-1)^2} = \frac{\frac{48-24}{21} - \ln(21)}{1}$$

$$= -1.90$$

b. Find the approximate change in y as x increases from 2 to 2 + p , where p is small.

[1]

$$\frac{dy}{dx} = \frac{\frac{6x^3-6x^2}{2x^3+5} - \ln(2x^3+5)}{(x-1)^2}$$

$$\frac{dy}{dx} = -1.9$$

$$\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$$

$$\delta y \approx -1.9p$$

4. $$f: x \to e^{3x} \text{ for } x \in \mathbb{R}$$

$$g: x \to 2x^2 + 1 \text{ for } x \geq 0$$

Solve $f'(x) = 6g''(x)$, giving your answer in the form $ln\ a$, where *a* is an integer.

[3]

$f'(x) = 3e^{3x}$

$g'(x) = 4x \rightarrow g''(x) = 4$

$3e^{3x} = 6 \times 4$

$3e^{3x} = 24$

$e^{3x} = 8$

$3x = \ln(8)$

$x = \frac{1}{3}\ln 8$

$= \frac{1}{3}\ln 2^3 = \ln 2$

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5. Two variables *x* and *y* are such that $y = \frac{lnx}{x^3}$ for $x > 0$.

a. Show that $\frac{dy}{dx} = \frac{1-3\,lnx}{x^4}$.

[3]

$\frac{dy}{dx} = \frac{\frac{1}{\cancel{x}} \times x^{\cancel{3}\,2} - 3x^2 \ln x}{(x^3)^2}$

$= \frac{x^2 - 3x^2 \ln x}{x^6} = \frac{1 - 3\ln x}{x^4}$

(shown)

b. Hence find the approximate change in *y* as *x* increases from e to e + *h*, where *h* is small.

[2]

$x = e$

$\delta x = h$

$\frac{dy}{dx} = \frac{1 - 3\ln x}{x^4}$

$= \frac{1-3}{e^4} = \frac{-2}{e^4}$

$\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$

$\delta y \approx \frac{-2}{e^4} h$

6. The variables x, y and u are such that $y = tan\, u$ and $x = u^3 + 1$.

a. State the rate of change of y with respect to u.

$$\frac{dy}{du} = \sec^2 u$$

[1]

b. Hence find the rate of change of y with respect to x, giving your answer in terms of x.

[4]

$y = \tan u$

$x = u^3 + 1$

$$\frac{dy}{dx} = ?$$

$$\frac{dy}{du} = \sec^2 u$$

$u = \sqrt[3]{x-1}$

$$\frac{dy}{du} = \sec^2 \sqrt[3]{x-1}$$

$$\frac{du}{dx} = \frac{1}{3}(x-1)^{-2/3}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \sec^2 \sqrt[3]{x-1} \times \frac{1}{3}(x-1)^{-2/3}$$

$$= \frac{\sec^2 \sqrt[3]{x-1}}{3\sqrt[3]{(x-1)^2}}$$

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7. Given that $y = \frac{\sin x}{\ln x^2}$, find an expression for $\frac{dy}{dx}$.

[4]

$$y' = \frac{\cos x \ln x^2 - \frac{1}{x^{\cancel{2}\,1}} \times 2\cancel{x} \times \sin x}{(\ln x^2)^2}$$

$$= \frac{\cos x \ln x^2 - \frac{2\sin x}{x}}{(\ln x^2)^2} = \frac{x\cos x \ln x^2 - 2\sin x}{x(\ln x^2)^2}$$

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8. Differentiate $\tan 3x \cos \frac{x}{2}$ with respect to x.

[4]

Let $y = \tan 3x \cos x/2$

$y' = 3\sec^2 3x \cos \frac{x}{2} - \frac{1}{2}\tan 3x \sin \frac{x}{2}$

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9. It is given that $y = \frac{ln(4x^2+1)}{2x-3}$.

a. Find $\frac{dy}{dx}$.

$$y' = \frac{\frac{1}{4x^2+1} \times 8x\,(2x-3) - 2\ln(4x^2+1)}{(2x-3)^2}$$

$$= \frac{\frac{16x^2-24x}{4x^2+1} - 2\ln(4x^2+1)}{(2x-3)^2}$$

[3]

b. Find the approximate change in *y* as *x* increases from 2 to 2+*h*, where *h* is small.

[2]

$$y' = \frac{\frac{16x^2-24x}{4x^2+1} - 2\ln(4x^2+1)}{(2x-3)^2}$$

$x=2$,
$\delta x = h$

$$y' = \frac{64-48}{17} - 2\ln(17) = -4.73$$

$$\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$$

$$-4.73h \approx \delta y$$

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10. It is given that $y = (1 + e^{x^2})(x + 5)$. $= x + 5 + xe^{x^2} + 5e^{x^2}$

a. Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 1 + e^{x^2} + 2x^2 e^{x^2} + 10x e^{x^2}$$

[3]

b. Find the approximate change in *y* as *x* increases from *0.5* to *0.5*+*p,* where *p* is small.

$x = 0.5$

$\delta x = p$

[2]

$$y' = 1 + e^{x^2} + 2x^2 e^{x^2} + 10x e^{x^2}$$

$$= 1 + e^{0.25} + 0.5e^{0.25} + 5e^{0.25}$$

$$= 1 + 6.5e^{0.25}$$

$$= 9.35$$

$$\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$$

$$9.35p \approx \delta y$$

c. Given that *y* is increasing at a rate of 2 units per second when *x* = 0.5, find the corresponding rate of change in *x*.

$$\frac{dy}{dt} = 2 \quad , x = 0.5$$

[2]

$$\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt} = \frac{1}{9.35} \times 2$$

$$= 0.214$$

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11. The equation of a curve is given by $y = xe^{-2x}$.

a. Find $\frac{dy}{dx}$.

$y' = e^{-2x} - 2xe^{-2x}$

[3]

b. Find the exact coordinates of the stationary point on the curve $y = xe^{-2x}$.

[2]

$\frac{dy}{dx} = 0$

$e^{-2x} - 2xe^{-2x} = 0$

$e^{-2x}(1 - 2x) = 0$

$e^{-2x} = 0$ or $1 - 2x = 0$

(reject) $2x = 1$

$x = \frac{1}{2}$

$y = \frac{1}{2}e^{-1} = \frac{1}{2e}$

c. Find, in terms of e, the equation of the tangent to the curve $y = xe^{-2x}$ at the point $(1, \frac{1}{e^2})$.

[2]

$$\frac{dy}{dx} = e^{-2x} - 2xe^{-2x}$$

$$= e^{-2} - 2e^{-2}$$

$$= -e^{-2}$$

$$y = -e^{-2}x + c$$

$$\frac{1}{e^2} = -e^{-2} + c$$

$$c = \frac{1}{e^2} + \frac{1}{e^2} = \frac{2}{e^2}$$

$$y = -\frac{1}{e^2}x + \frac{2}{e^2}$$

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12. Given that $y = 2sin\,3x + cos\,3x$, show that $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 3y = ksin\,3x$, where k is a constant to be determined. [5]

$$\frac{dy}{dx} = 6\cos 3x - 3\sin 3x$$

$$\frac{d^2y}{dx^2} = -18\sin 3x - 9\cos 3x$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + 3y = k\sin 3x$$

$$= -18\sin 3x \cancel{- 9\cos 3x} \cancel{+ 6\cos 3x} - 3\sin 3x + 6\sin 3x \cancel{+ 3\cos 3x}$$

$$= -15\sin 3x$$

$$\therefore k = -15$$